JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 89, No. 1, pp. 115-127, APRIL 1996

Vector Linear Programming in Zero-Sum Multicriteria Matrix Games

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Communicated by P. L. Yu

Abstract. In this paper, a multiple-objective linear problem is derived from a zero-sum multicriteria matrix game. It is shown that the set of efficient solutions of this problem coincides with the set of Pareto-optimal security strategies (POSS) for one of the players in the original game. This approach emphasizes the existing similarities between the scalar and multicriteria matrix games, because in both cases linear programming can be used to solve the problems. It also leads to different scalarizations which are alternative ways to obtain the set of all POSS. The concept of ideal strategy for a player is introduced, and it is established that a pair of Pareto saddle-point strategies exists if both players have ideal strategies. Several examples are included to illustrate the results in the paper.

Key Words. Game theory, multicriteria games, Pareto-optimal security strategies, vector linear programming, scalarization methods.

1. Introduction

Recently, much attention has been paid to game problems in which the payoff is a multiple noncomparable criteria vector (Refs. 1–3). One of the reasons is that this approach represents better real-world applications of game theory (Refs. 4–5). In fact, each competitive situation that can be modeled as a scalar zero-sum game has its counterpart as a multicriteria zero-sum game when more than one scenario has to be compared simultaneously. In these situations, once the same strategy has to be used in different scenarios, conflicting interests appear between different decision-makers as

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well as within each individual. For instance, the production policies of two firms which are competing for a market can be seen as a scalar game. However, when they compete simultaneously in several markets, the multicriteria approach has to be adopted.

When cooperation is not allowed in the games, in general there is not a total order among the vector payoffs. Hence, comparing the payoff obtained by the players in multicriteria games is much more difficult than comparing them in scalar games, and the classical solution concepts are not applicable.

For this reason, new solution concepts have been proposed in recent years (Refs. 2–3 and 6–7), and have been compared with the existing ones. Particularly, the concept of Pareto-optimal security strategy (POSS) becomes very important in order to solve a multicriteria game (Ref. 2).

In Ref. 1, POSS in a zero-sum multicriteria matrix game were obtained for a player by scalarization of the original game. The main result in the paper states, using many intermediate steps, that an extension of the set of all security level vectors is a polyhedral set. This implies that the saddle-point solution of a game with strictly positive scalarization is both a necessary and a sufficient condition to obtain POSS.

In this paper, we obtain the same characterization as a particular instance of a general approach in an alternative simpler way. Using the powerful tools of vector linear programming, we can obtain all POSS either as efficient solutions of multiple-objective linear problems, or as solutions of parametric linear problems, or as minimax solutions of scalar games, or as solutions of weighted minimax problems. These approaches lead us, by means of a finite set of POSS, to obtain all POSS. The POSS of this finite set are the efficient extreme solutions for a multiple-objective linear problem.

The paper is organized as follows. Section 2 states the general setting where we formulate the multicriteria matrix games and recall the concept of POSS. Section 3 defines a multiple-objective linear problem (MOLP) whose efficient solutions coincide with the POSS. In addition, a necessary and sufficient condition for the existence of Pareto saddle-point strategies is obtained, and some examples showing these concepts and the solution method proposed in the paper are included. Associated with this MOLP, there exist several scalarizations whose solutions have the same property. Section 4 is devoted to the introduction and interpretation of the different parameterizations proposed. Finally, Section 5 concludes the paper.

2. Model and Previous Results

We consider a two-person finite game in normal form (matrix game) with a payoff matrix A with n rows and m columns. Each element a_{ij} of the matrix A is a k-dimensional vector $(a_{ij}(1), \ldots, a_{ij}(k))$. We define individual matrices of dimension $n \times m$ as

$$A(l) = \{a_{ij}(l)\}, \qquad l = 1, \ldots, k.$$

The players are represented by P1 (the minimizer, who chooses rows) and P2 (the maximizer, who chooses columns). As usual, the mixed strategy spaces for players P1 and P2 are

$$\Gamma^1 = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0, i = 1, \ldots, n \right\}, \tag{1}$$

$$\Gamma^2 = \left\{ y \in \mathbb{R}^m : \sum_{j=1}^{m} y_j = 1, y_j \geq 0, j = 1, \ldots, m \right\}. \tag{2}$$

We remark that the pure strategies for both players are the extreme points of Γ^1 and Γ^2. It is easy to see that the extreme points of Γ^1 [respectively, Γ^2] are $e^i \in \mathbb{R}^n$, $i = 1, \ldots, n$ [$e^j \in \mathbb{R}^m$, $j = 1, \ldots, m$], where e^i is a vector with 1 in the ith coordinate and zero everywhere else.

Choosing $x \in \Gamma^1$ and $y \in \Gamma^2$ implies that the expected payoff of the game is

$$v(x, y) = x^t A y = [v_1(x, y), \ldots, v_k(x, y)], \tag{3}$$

where

$$v_l(x, y) = x^t A(l) y, \qquad l = 1, \ldots, k. \tag{4}$$

In the sequel, the transpose operator t will be omitted when its use is clear.

Every strategy $x \in \Gamma^1$ [respectively, $y \in \Gamma^2$] defines security levels $\bar{v}_l(x)$ [respectively, $\underline{v}_l(y)$] as the payoffs with respect to every criterion v_l, $l = 1, \ldots, k$, when P2 [respectively, P1] bets to minimize [respectively, maximize] the criteria (Ref. 2). Hence,

$$\bar{v}_l(x) = \max_{y \in \Gamma^2} v_l(x, y), \qquad l = 1, \ldots, k, \tag{5}$$

$$\underline{v}_l(y) = \min_{x \in \Gamma^1} v_l(x, y), \qquad l = 1, \ldots, k, \tag{6}$$

and the security levels are k-tuples denoted by

$$\bar{v}(x) = [\bar{v}_1(x), \ldots, \bar{v}_k(x)], \tag{7}$$

$$\underline{v}(y) = [\underline{v}_1(y), \ldots, \underline{v}_k(y)]. \tag{8}$$

It must be remarked that, for a given strategy x for P1, the security levels $\bar{v}_l(x)$, $l = 1, \ldots, k$, might be obtained for the player P2 by different strategies. Using our notation, we now introduce the definition of POSS given in Ref. 2.

Definition 2.1. A strategy $x^* \in \Gamma^1$ is a Pareto-optimal security strategy (POSS) for P1 iff there is no $x \in \Gamma^1$ such that $\bar{v}(x^*) \geq \bar{v}(x)$, $\bar{v}(x^*) \neq \bar{v}(x)$.

Similarly, one can define POSS for P2.

We now introduce the general multiple-objective linear problem and state some properties about it:

$$\begin{array}{lll} \text{(MOLP)} & \min & Cx, \\ & \text{s.t.} & Ax \leq b, \\ & & x \geq 0, \\ & \text{where} & C \in \mathbb{R}^{p \times q},\ A \in \mathbb{R}^{r \times q},\ b \in \mathbb{R}^r. \end{array}$$

We can consider the set of efficient vectors as the solution set for this problem. The efficient solutions (Ref. 8) are usually defined as follows.

Definition 2.2. A feasible solution x^* of (MOLP) is an efficient solution iff there does not exist x feasible, such that $Cx \leq Cx^*$, $Cx \neq Cx^*$.

Associated with the problem (MOLP) there is a family of parametric linear problems,

$$\begin{array}{lll} (\mathrm{P}(w)) & \min & w^t Cx, \\ & \text{s.t.} & Ax \leq b, \\ & & x \geq 0, \\ & & w \in W^0 = \left\{ w \in \mathbb{R}^p : w > 0,\ \sum_{i=1}^{p} w_i = 1 \right\}. \end{array}$$

The equivalence between both problems is well known and is given by the following theorem.

Theorem 2.1. x^* is an efficient solution of (MOLP) iff $\exists w^0 \in W^0$ such that x^* is an optimal solution of $(\mathrm{P}(w^0))$.

See Ref. 9 for a proof. The characterization of POSS that we propose is based on this theorem.

3. Determination of POSS

As was pointed out by Ghose in Ref. 1, although a way to determine all POSS of a player is through scalarization, other methods must not be discarded. We consider an alternative approach to the one proposed in Ref.

1, which simplifies very much the proofs of the characterizations given. The main goal of this paper is to prove that a necessary and sufficient condition for a strategy to be a POSS is to be an efficient solution of a MOLP.

Recall that, for a strategy $x \in \Gamma^1$, the lth security level for player P1 was defined in (5). This is a scalar linear program, so it has an optimal solution among the extreme points of Γ^2; hence,

$$\bar{v}_l(x) = \max_{j=1,\ldots,m} \sum_{i=1}^{n} x_i a_{ij}(l), \tag{9}$$

or in vector form

$$\bar{v}_l(x) = \max xA(l). \tag{10}$$

Now, we introduce the following multiple-objective linear program, called the multicriteria linear game problem (MLGP)

$$\begin{aligned} \text{(MLGP)} \quad \min \quad & v_1, \ldots, v_k, \\ \text{s.t.} \quad & xA(l) \leq (v_l, \ldots, v_l), \qquad l = 1, \ldots, k, \\ & \sum_{i=1}^{n} x_i = 1, \\ & x \geq 0, \qquad v \in \mathbb{R}^k. \end{aligned}$$

The main theorem of this section is the following.

Theorem 3.1. A strategy $x^* \in \Gamma^1$ is a POSS and $v^* = (v_1^*, \ldots, v_k^*)$ is its security level vector iff (v^*, x^*) is an efficient solution of problem (MLGP).

Proof. Let x^* be a POSS. Then, there is no $x \in \Gamma^1$ such that $\bar{v}(x) \leq \bar{v}(x^*)$, $\bar{v}(x) \neq \bar{v}(x^*)$ by definition. From (10), this is equivalent to

$$(\max xA(1), \ldots, \max xA(k)) \leq (\max x^*A(1), \ldots, \max x^*A(k)),$$
$$(\max xA(1), \ldots, \max xA(k)) \neq (\max x^*A(1), \ldots, \max x^*A(k));$$

hence, x^* is an efficient solution of the problem

$$\min_{x \in \Gamma^1} (\max xA(1), \ldots, \max xA(k));$$

now, using the usual transformation of minmax into linear problems (Ref. 10), the above minmax problem is equivalent to

$$\begin{aligned} \text{(MLGP)} \quad \min \quad & v_1, \ldots, v_k, \\ \text{s.t.} \quad & xA(l) \leq (v_l, \ldots, v_l), \qquad l = 1, \ldots, k, \\ & \sum_{i=1}^{n} x_i = 1, \\ & x \geq 0, \qquad v \in \mathbb{R}^k. \end{aligned}$$

Conversely, suppose that an efficient solution (v^*, x^*) of (MLGP) is not a POSS. Then, there exists $\bar{x} \in \Gamma^1$ such that

$$(\max \bar{x}A(1), \ldots, \max \bar{x}A(k)) \leq (\max x^*A(1), \ldots, \max x^*A(k)),$$

$$(\max \bar{x}A(1), \ldots, \max \bar{x}A(k)) \neq (\max x^*A(1), \ldots, \max x^*A(k)).$$

Taking $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_k)$, where

$$\bar{v}_l = \max \bar{x}A(l), \qquad l = 1, \ldots, k,$$

the vector $(\bar{v}, \bar{x})$ is a feasible solution of (MLGP) dominating (v^*, x^*). This is a contradiction. □

The theorem above is very important for several reasons. First, just as linear programming can be used to find the value and the optimal strategies for any scalar two-person zero-sum matrix game, vector linear programming can also be applied to find jointly all POSS for a player and its security levels. This emphasizes the similarity existing between both problems.

Secondly, it must be remarked that, as usual in (MOLP), obtaining the efficient extreme solutions suffices to generate all POSS. Procedures for computing (characterizing) all maximally efficient facets have been proposed by different authors (Refs. 11–14). Thus, existing algorithms valid for the determination of efficient solutions of (MOLP), as for instance ADBASE (Ref. 15), can be applied to calculate all POSS.

Example 3.1. Consider the following payoff matrix proposed by Ghose in Ref. 1:

$$A = \begin{bmatrix} (1, 3) & (2, 1) \\ (3, 1) & (1, 2) \\ (1, 1) & (3, 3) \end{bmatrix}.$$

Using the software package ADBASE (Ref. 15), we have obtained the efficient extreme solutions of the following problem:

$$\begin{aligned}
\min \quad & (v_1, v_2), \\
\text{s.t.} \quad & x_1 + 3x_2 + x_3 \leq v_1, \\
& 2x_1 + x_2 + 3x_3 \leq v_1, \\
& 3x_1 + x_2 + x_3 \leq v_2, \\
& x_1 + 2x_2 + 3x_3 \leq v_2, \\
& x_1 + x_2 + x_3 = 1, \\
& x \geq 0, \qquad v_1 \in \mathbb{R}, \qquad v_2 \in \mathbb{R}.
\end{aligned}$$

The extreme efficient solutions are

$$(\bar{v}(x^1), x^1) = (9/5, 9/5, 2/5, 2/5, 1/5),$$

$$(\bar{v}(x^2), x^2) = (7/3, 5/3, 1/3, 2/3, 0),$$

$$(\bar{v}(x^3), x^3) = (5/3, 7/3, 2/3, 1/3, 0),$$

and the following set gives all POSS for player P1:

$$\mathrm{ch}\{(2/5, 2/5, 1/5), (1/3, 2/3, 0)\} \bigcup \mathrm{ch}\{(2/5, 2/5, 1/5), (2/3, 1/3, 0)\},$$

where $\mathrm{ch}\{a, b\}$ is the convex hull of the vectors a, b.

Finally, Theorem 3.1 leads us to characterize when a pair of POSS are Pareto saddle-point strategies. A strategy pair $x \in \Gamma^1$ and $y \in \Gamma^2$ is said to be a pair of Pareto saddle-point strategies if $\bar{v}(x) = \underline{v}(y)$; see Ref. 6. In multiple-objective programming, an ideal solution is a feasible solution which minimizes simultaneously all the objectives. Considering the framework of multiple criteria games, then we say that x^* is an ideal strategy for player P1 if x^* minimizes $\bar{v}_l(x)$, $\forall l = 1, \ldots, k$ (analogously for P2). However, the existence of an ideal strategy for a player does not imply the existence of Pareto saddle-point strategies for the multiple criteria game, because the security levels for each scalar game $A(l)$, $l = 1, \ldots, k$ may be achieved with different strategies by the other player.

Corollary 3.1. $x^* \in \Gamma^1$ and $y^* \in \Gamma^2$ are a pair of Pareto saddle-point strategies for player P1 and P2 if and only if x^* and y^* are ideal strategies for player P1 and P2, respectively.

As each ideal strategy is a saddle-point strategy for every scalar game $A(l)$, $l = 1, \ldots, k$, the corollary above is equivalent to Theorem 4.1 in Ref. 2.

Some examples are now considered.

Example 3.2. Consider the following game matrix, proposed by Ghose and Prasad in Ref. 2:

$$A = \begin{bmatrix} (2, 3) & (3, 2) \\ (4, 1) & (2, 3) \end{bmatrix}.$$

There are Pareto-optimal saddle-point strategies for the game, which for player P1 is $(2/3, 1/3)$ and for player P2 is $(1/3, 2/3)$. It must be remarked that both strategies are ideal strategies for players P1 and P2, respectively.

Example 3.3. Consider the payoff matrix

$$A=\begin{bmatrix}(1,3) & (3,2)\\ (6,1) & (2,3)\end{bmatrix}.$$

In this game, the player P1 has an ideal strategy (2/3, 1/3). However, the POSS for player P2 are given by the convex hull of the strategies (1/3, 2/3) and (1/6, 5/6). Obviously, in this case, there does not exist an ideal strategy for player P2, and there is not a pair of Pareto saddle-point strategies.

4. Scalarization Methods

The characterization given by Theorem 3.1 permits several ways of scalarization for the solution of a multicriteria game. In particular, the scalarization proposed by Ghose in Ref. 1 is obtained as a particular case.

The first scalarization that we consider is given through a scalar linear problem associated with (MLGP):

$$\begin{aligned}
&(\mathrm{P}(\lambda)) \quad \min && \sum_{l=1}^{k} \lambda_l v_l,\\
&\qquad\qquad \text{s.t.} && xA(l)\leq (v_l,\ldots,v_l), \qquad l=1,\ldots,k,\\
&&& \sum_{i=1}^{n} x_i=1,\\
&&& x\geq 0, \qquad v\in\mathbb{R}^k,\\
&&& \lambda\in\Lambda^0=\left\{\lambda\in\mathbb{R}^k:\lambda>0,\ \sum_{l=1}^{k}\lambda_l=1\right\}.
\end{aligned}$$

Theorem 4.1. A strategy $x^*\in\Gamma^1$ is a POSS and v^* is its associated security level vector iff there exists $\lambda^*\in\Lambda^0$ such that (v^*, x^*) is an optimal solution of problem $(\mathrm{P}(\lambda))$.

The proof follows by using the characterization of POSS given in Theorem 3.1 and by applying the scalarization of Theorem 2.1.

Every component λ_l of the parameter $\lambda=(\lambda_1,\ldots,\lambda_k)\in\Lambda^0$ can be interpreted as the relative importance the player P1 assigns to the scalar game with matrix $A(l)$. Thus, given a parameter λ, it induces a weak order among all POSS by means of the scalarization of their security levels.

Besides, each optimal strategy x^* is associated with a polyhedral set $\Lambda(x^*)\subset\Lambda^0$, so that (v^*, x^*) is an optimal solution of $(\mathrm{P}(\lambda))$, $\forall\lambda\in\Lambda(x^*)$.

This set can be easily obtained, because

$$\Lambda(x^*) = \{\lambda \in \Lambda^0 : \lambda'(Z - C)(x^*) \leq 0\},$$

where $(Z-C)(x^*)$ is the reduced cost matrix associated with the efficient solution x^* (Ref. 15). The set $\Lambda(x^*)$ could be seen as a sensitivity region, because small changes in the parameters do not change the optimal strategy as soon as this region has nonempty relative interior. It is interesting to remark that the above parametric regions associated with the extreme solutions are usually bigger than those associated with the rest of POSS. These sets induce a partition in Λ^0 and in the set of all POSS for player P1.

Let Γ^1_{sp} be the set of all Pareto-optimal security strategies for P1 (similarly Γ^2_{sp} for P2), i.e.,

$$\Gamma^1_{sp} = \{(\bar{v}(x), x) : x \in \Gamma^1, x \text{ is a POSS}\};$$

let $\Gamma_{sp}(x)$ be the set of optimal solutions of $(P(\lambda))$, $\forall \lambda \in \Lambda(x)$, and ext(MLGP) the set of efficient extreme solutions of (MLGP). From the above discussion, one gets the following theorem.

Theorem 4.2. $\Gamma^1_{sp} = \bigcup_{x \in \text{ext(MLGP)}} \Gamma_{sp}(x)$.

Proof. Let x be a POSS. Then, from Theorem 4.1, there exists $\lambda^0 \in \Lambda^0$ such that x is an optimal solution of $(P(\lambda^0))$. Now, as $(P(\lambda^0))$ is a linear program, it has at least an extreme point x^* of its feasible region which is also an optimal solution. Hence, λ^0 belongs to $\Lambda(x^*)$, which implies that $x \in \Gamma_{sp}(x^*)$.

Conversely, let $x^0 \in \Gamma_{sp}(x)$ for some $x \in$ ext(MLGP). Then, x^0 is an optimal solution of $(P(\lambda))$ for all $\lambda \in \Lambda(x) \subseteq \Lambda^0$. Hence applying Theorem 4.1, it follows that x^0 is a POSS. □

The following example illustrates the theorem above.

Example 4.1. Consider the game described in Example 3.1. The goal is to obtain the decomposition given by Theorem 4.2 of the set Γ^1_{sp} and the partition of Λ^0 in the sets $\Lambda(x^i)$, $i=1, 2, 3$, which give the sensitivity regions associated to each efficient extreme strategy x^i, $i=1, 2, 3$.

The sensitivity regions $\Lambda(x^i)$, $i=1, 2, 3$, are given by $\lambda(Z-C)(x^i) \leq 0$. In this example, as soon as

$$\Lambda^0 = \{(\lambda, 1-\lambda) : \lambda \in [0, 1]\},$$

the sets $\Lambda(x^i)$ can be described with only one parameter, that is

$$\Lambda(x^i) = \{\lambda : \lambda \in [0, 1], (\lambda, 1-\lambda)(Z-C)(x^i) \leq 0\},$$

and they correspond to

$$\Lambda(x^1) = \mathrm{ch}\{1/5, 4/5\},$$

$$\Lambda(x^2) = \mathrm{ch}\{0, 1/5\},$$

$$\Lambda(x^3) = \mathrm{ch}\{4/5, 1\}.$$

Finally, using inside these sets the usual sensitivity analysis of linear programming, one can obtain the sets $\Gamma_{sp}(x^i)$, $i = 1, 2, 3$,

$$\Gamma_{sp}(x^1) = \{(2/5, 2/5, 1/5)\},$$

$$\Gamma_{sp}(x^2) = \mathrm{ch}\{(2/5, 2/5, 1/5), (2/3, 1/3, 0)\},$$

$$\Gamma_{sp}(x^3) = \mathrm{ch}\{(2/5, 2/5, 1/5), (1/3, 2/3, 0)\}.$$

Example 4.2. In this example, we consider the games proposed in Examples 3.2 and 3.3. Our aim is to obtain the decomposition given by Theorem 4.2 and the partition of Λ^0.

As in these examples, there exists an ideal solution $x^* = (2/3, 1/3)$ for P1, the sensitivity region $\Lambda(x^*) = [0, 1]$, and the set $\Gamma_{sp}(x^*) = \{(2/3, 1/3)\}$.

The importance of POSS as a solution concept was pointed out in Refs. 1–2. Obtaining the sets of POSS for the players is useful if one wants to check the existence of solutions based upon the idea of security levels. Another way to obtain those sets consists of defining scalar criterion games and proving that the minimax [respectively, maxmin] solutions of these games are also POSS for the corresponding players.

As a consequence of Theorem 3.1, we give also a very easy proof of the main result in Ref. 1; that is, being a saddle-point solution of a zero-sum game is a necessary and sufficient condition for a strategy to be a POSS.

Let x, y be two POSS for players P1 and P2, respectively, i.e., $x \in \Gamma^1_{sp}$, $y \in \Gamma^2_{sp}$. As the vectors of security levels $\underline{v}(y)$, $\bar{v}(x)$ must dominate every payoff, it holds that

$$\underline{v}(y) \leq xAy \leq \bar{v}(x), \tag{11}$$

which leads to the characterization of all POSS by means of scalar criterion games.

Consider a game where the player J1 chooses his strategies in Γ^1, the player J2 chooses k different strategies $(y^1, \ldots, y^k)$ in Γ^2, and the payoff function is given by

$$v(x, \underline{y}, \lambda) = \sum_{l=1}^{k} \lambda_l x A(l) y^l, \tag{12}$$

where $\underline{y}=(y^1, \ldots, y^k)$, being $y^l \in \Gamma^2$, $l=1, \ldots, k$, and $\lambda=(\lambda_1, \ldots, \lambda_k) \in \Lambda^0$. Hereafter, we call $(J(\lambda))$ this game.

As soon as $(J(\lambda))$ is a scalar game, the usual definition for minimax strategy (Ref. 10) holds for both players. Then, we can state the equivalence between POSS and the minimax strategy for $J(\lambda)$.

Theorem 4.3. A strategy $x^* \in \Gamma^1$ is POSS iff there exists a $\lambda \in \Lambda^0$ such that x^* is a minimax solution of $(J(\lambda))$.

Proof. Using the fact that $(J(\lambda))$ is a scalar zero-sum matrix game, x^* is a minimax strategy for $(J(\lambda))$ iff it is an optimal solution of

$$\min_{x \in \Gamma^1} \max_{(y^1, \ldots, y^k) \in (\Gamma^2)^k} \sum_{l=1}^{k} \lambda_l x A(l) y^l,$$

and iff it is an optimal solution of

$$\min_{x \in \Gamma^1} \sum_{l=1}^{k} \lambda_l \max_{y^l \in (\Gamma^2)} x A(l) y^l = \min_{x \in \Gamma^1} \sum_{l=1}^{k} \lambda_l \bar{v}_l(x).$$

Now, as the last formulation is equivalent to $(P(\lambda))$, then x^* is a minimax strategy for $(J(\lambda))$ iff $(\bar{v}(x^*), x^*)$ is an optimal solution of $(P(\lambda))$. □

Although this result was already known (see Theorem 3.3 in Ref. 1), the proof above is simpler than the one presented in Ref. 1. That paper showed that a certain set, which is the extension of the set of security level vectors, is convex and polyhedral. The new proof, based on the powerful tools of the (MOLP), follows easily from the equivalence between the scalar problem $(P(\lambda))$ and the game $(J(\lambda))$.

From Theorem 4.2, it follows that, given $x^* \in \Gamma^1_{sp}$, then $(\bar{v}(x^*), x^*)$ is an optimal solution of the problem $(P(\lambda))$, $\forall \lambda \in \Lambda(x^*)$. The theorem above asserts that the same x^* is also a minimax strategy for the game $(J(\lambda))$, $\forall \lambda \in \Lambda(x^*)$. However, obtaining all POSS by means of the games $(J(\lambda))$ leads us to the problem of how to determine the minimal number of scalarization coefficients to generate all POSS, while using the (MOLP) approach, it is only necessary to identify the set of efficient extreme solutions.

Moreover, it is possible to give another scalarization to get all POSS based on the concept of weighted minimax problem. For each

$$w \in W = \{w \in \mathbb{R}^k : w > 0\},$$

consider the following problem:

$$(\mathrm{WMP})(w)) \quad \min_{x \in \Gamma^1} \max_{1 \le l \le k} w_l \bar{v}_l(x), \tag{13}$$

or equivalently,

$$\begin{array}{lll}\min\limits_{x\in\Gamma^1} & z, & \\ \text{s.t.} & xA(l)\leq(v_l,\ldots,v_l), & l=1,\ldots,k, \\ & w_l\cdot v_l\leq z, & l=1,\ldots,k, \\ \text{where} & z\in\mathbb{R}. & \end{array}$$

The following theorem states that the solution of (WMP(w)) with strictly positive w (i.e., $w>0$) is both a necessary and sufficient condition for a strategy to be a POSS for P1.

Theorem 4.4. A strategy $x^*\in\Gamma^1$ is POSS for P1 in the original multicriteria game if and only if $(\bar{v}(x^*), x^*)$ is an optimal solution to (WMP(w)) with $w\in W$.

Proof. From Theorem 1 in Ref. 16, $(\bar{v}(x^*), x^*)$ is an efficient solution to (MLGP) iff there is a $w^0\in W$ such that $(\bar{v}(x^*), x^*)$ is an optimal solution to (WMP(w^0)). Now, from Theorem 3.1, this is equivalent to be a POSS for player P1 in the original game. This concludes the proof. □

5. Conclusions

In earlier works, the concept of Pareto-optimal security strategy (POSS) has been introduced and a necessary and sufficient condition was proved to characterize POSS for a player in a two-person zero-sum matrix game. That method requires the specific determination of several coefficients which are the basis of the scalarization proposed and, as is suggested in Ref. 1, that is not an easy task. In this paper, new necessary and sufficient conditions based on vector linear programming are proved which characterize all POSS. This methodology leads us to obtaining all POSS, once the efficient set of a particular multiple-objective linear program is obtained. Several scalarizations are proposed which are direct consequences of the main result. In particular, the above-mentioned necessary and sufficient condition (Ref. 1) is obtained as a special instance.

Finally, it should be remarked that the proposed approach states the relationship existing in the solution method between the scalar zero-sum matrix games and the multicriteria zero-sum matrix games. The first one can be solved using linear programming and the second one using vector linear programming.

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